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CONSISTENCY & INCONSISTENCY OF DOMINATION NUMBERS IN DIGRAPHS

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ABSTRACT
This paper presents the application of domination in digraph which is useful in designing a graph model for fault tolerant computing system. The effect of $\gamma\ (D)$ when D is modified by deletion of a vertex or deletion of an edge were investigated by several authors. In this paper we discuss the analogous problem for directed graphs. Also find the bondage number for digraph.
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Keywords- *Graph, Digraph, Domination Set, Domination Number, Bondage Number*

I. INTRODUCTION

In the topological design of a network an important consideration is fault tolerance, that is, the ability of the network to provide service even when it contains a faulty component or components. The behavior of a network in the presence of a fault can be analyzed by determining the effect that deletion of an arc (link failure) or a vertex (process or failure) from it sunder lying digraph D has on the fault tolerance criterion. For example a γ- set in D represent the minimum number of processors that can communicate directly with all other processors in the system. For file servers if it is essential to have this property and that the number of processors designated as file servers be limited, then the domination number of D is the fault-tolerance criterion. In this example, it is most γ(D) does not increase when D is modified by removing a vertex or an edge. From another prospective network can be made fault tolerant by providing redundant communication links (adding edges). Hence we examine the effects on γ(D) when D is modified by deleting a vertex or deleting an edge.

Definition
Let D (V, A) be a digraph. We proceed to consider the effect of deletion of a vertex or deletion of an edges of D.

Examples
(i) Let D be the complete symmetric digraph on n vertices where $n \geq 5$. Clearly $\gamma(D) = 1$ and $\gamma(D - v) = 1$ for every $v \in V(D)$ Further $\gamma\ (D - e) = 1$ for every arc $e \in A(D)$. Hence in this example removal of any vertex or removal of any arc does not change the domination number of D.
(ii) Consider the star K_{1n} which is oriented in such a way that all pendant vertices have outdegree zero. Then the central vertex u has outdegree n and $\{u\}$ is dominating set of the digraph. Hence $\gamma\ (D) = 6$. Further $\gamma(D - v) = 1$ for every pendant vertex v of D and $\gamma(D - u) = n$ where u is the central vertex of D. Also for any arc e of $D, \gamma(D - e) = 5$.
(iii) Consider the directed cycle C_4. Here $\gamma = 2, \gamma(C_4 - v) = 2$ and $\gamma(C_4 - e) = 2$ for every vertex v and for every arc e of the directed cycle.
(iv) Consider the directed graph D given in figure 1.

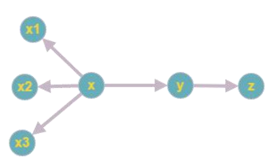

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Fig .1

Here $\gamma(D) = 2$, $\gamma(D - x) = 4$, $\gamma(D - z) = 1$ and $\gamma(D - y) = 2$
(v) Consider the digraph D given in figure 2.

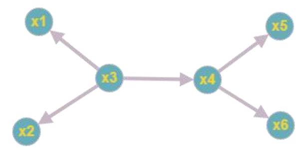

Fig. 2

Here $\gamma(D) = 2$, $\gamma(D - x_3x_4) = 2$, $\gamma(D - x_4x_5) = 3$.

The above examples show that removal of a vertex may increase, decrease or unalter the domination number of a digraph and removal of an arc may increase or unalter the domination number of a digraph.

Hence we partition of vertices of a digraph D into three sets.

Let $V = V^0 U\, V^+ U\, V^-$ where

$V^0 = \{v \epsilon V: \gamma(D - v) = \gamma(D)\}$
$V^+ = \{v \epsilon V: \gamma(D - v) > \gamma(D)\}$
$V^- = \{v \epsilon V: \gamma(D - v) < \gamma(D)\}$

Similarly the edge set can be partitioned into

$E^0 = \{uv \epsilon E: \gamma(D - uv) = \gamma(D)\}$
$E^+ = \{uv \epsilon E: \gamma(D - uv) > \gamma(D)\}$

For a digraph for which the domination number changes when an arbitrary vertex is removed, we have $V = V^- \cup V^+$. It can be shown that $V^{"}$ is never empty for a directed tree.

Theorem 1

For any directed tree$T$, with $n \geq 2$, there exists a vertex $v \in V$ such that $\gamma(T - v) = \gamma(T)$.

Proof

Clearly the result is true for a directed K_2. Let T_0 be the underlying graph of T. Now assume that T_0 has at least one vertex y, with $d\,(y) \geq 2$. Choose y so that it is adjacent to at least one pendant vertex and at most one nonpendant vertex. We consider two cases.

Case (i) y is adjacent to at least two pendant vertices x_1, x_2in T_n.

Now if $(y, x1)$ and $(y, x2)$ are arcs in the directed tree T, then y is in every γ - set of $T$and $\gamma(T - x_1) = \gamma(T)$. [Refer fig. 3]

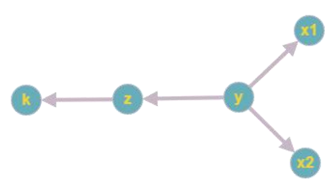

Fig. 3

(x_1, y) and (x_2, y) are arcs in T, then both x_1 and x_2 are in every γ - set and hence $\gamma(T - y) = \gamma(T)$. (Refer fig. 4)

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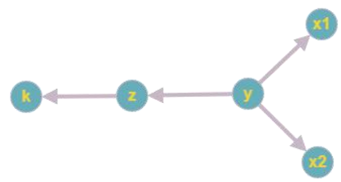

Fig .4

Suppose that (y, x_1) and (x_2, y) are arcs in T. If y belongs to some $\gamma(T)$ set. Then any $\gamma(T-y)$ set must contain x_1 instead of y and hence $\gamma(T-y) = \gamma(T)$. [Refer fig. 5]

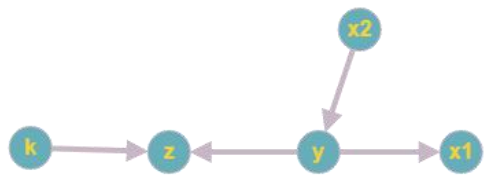

Fig. 5

If y is not included in some $\gamma(T)$set S, then $x_1 \in S$ and hence $\gamma(T-y) = \gamma(T)$.
[Refer fig. 6]

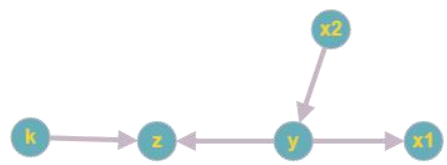

Fig. 6

Case (ii) y is adjacent to just one end vertex say x in T_0. Hence $d_{r0}(y) = 5$. If x has outdegree 0 in T, then $\gamma(T-y) = \gamma(T)$. If x has indegree 0 in T, it is included in every γ - set of T. So there exists a γ - set not containing y. Therefore $\gamma(T-x) = \gamma(T)$.

Deleting a vertex can expand the domination number by greater than one, but can reduce it by at most one. Deleting the centre of an outward star expand the domination number by $(n-2)$ and deleting an end vertex of an inward star reduces it by one.

The directed path $P_{2k-1}, K > 1$ is another example of a digraph for which the deletion of an end vertex reduces the domination number by one.
If S is a γ - set, then deleting any vertex in $V - S$ cannot expand the domination number, and so $|V^+| \le \gamma(D)$. Obviously every isolated vertex is in V.

Theorem 2
A vertex v of a digraph $D = (V, A)$ is in V if and only if pon $[v, S] = \{v\}$ for some γ - set S containing v or there exist $u, w \in S$ such that $w \in O\ (u), v \in O\ (w)$ and pon $[w, S] = \{v\}$ if S does not contain v.

Proof
Let $v \in V$ and R be some γ - set of $D - v$. Then $S = R \cup \{v\}$ is a γ - set of D. If R contains an element u such that $v \in O\ (u)$ then R itself is a dominating set of D, a contradiction to the assumption that $v \in V$. Therefore pon$[v, S] =$

$\{v\}$.If $v \in O\ (w), w \notin R$, then $S = R \cup (w)$ is a γ - set of D. If R contains an element u such that $v \in O\ (u)$, then R itself is a γ - set of D, a contradiction. Therefore pon $[w, S] = \{v\}$. Since R is a dominating set of $D - v$, there exists $u \in R$ such that $w \in O\ (u)$.

Conversely if pon $[v, S] = \{v\}$, $v \in S$, then $S - \{v\}$ is a dominating set of $D - v$ or if $u, w \in S$ such that $w \in O\ (u), v \in O\ (w)$ and pon $[n, S] = \{v\}$, then $S - \{w\}$ dominates $D - v$. Thus in either case $v \in V$.

Theorem 3
Let $D = (V, A)$ be a digraph. A vertex v belongs to V if and only if (a) v is not an isolate and v is in every γ - set and (b) no subset $S \subseteq V - I[v]$ of cardinality $\gamma(D)$ dominates $D - v$.

Proof
Let $v \in V$. Then $\gamma(D - v) > \gamma(D)$.
Clearly v is not an isolate of D. If there is a $\gamma(D$ - set $R \subseteq V\ (D)$ not containing v then R is a dominating set of $D - v$, so that $\gamma(D - v) \leq \gamma(D)$, a contradiction. Hence v is in every $\gamma(D)$ set. If there exists $S \subseteq V - I[v]$, with cardinality $\gamma(D)$, dominating $D - v$, then $\gamma(D - v) \leq \gamma(D)$, a contradiction. Therefore no such S exists.

Conversely assume that (a) and (b) hold. Since there is no subset $S \subseteq V - I[v]$ of cardinality $\gamma(D)$ dominating $D - v$, $\gamma(D - v)$, $\gamma(D)$Hence $v \in V'$.

Theorem 4
Let D be any digraph and $v \in V$. Then for any γ - set S of D containing$v$, the set pon $[v, S]$ has more than one vertex and no vertex of the set dominates all other vertices of the set.

Proof
We know that each $v \in V'$ is not an isolated vertex and is in every γ - set. Now pon $[v, S] \neq \phi$, for otherwise any $u \in I\ (v)$, $(S - \{v\}) \cup \{u\}$ is a dominating set. Let $w \in pon\ [v, S]$. If $pon\ [v.S]$ contains w alone or if w dominates all other vertices in $pon[v, S]$ then $(S - \{v\}) \cup (u)$ will be a dominating set of $D - v$, a contradiction. Hence the result.

Theorem 5
If $x \in V'$and $y \in V$, then$(x, y) \notin A\ (D)$.

Proof
Let $(x, y) \in A\ (D)$ and S_y be a dominating set of cardinality $\gamma(D) - 6$. If S_y contains x, then it dominates D, contradicting that $\gamma(D)$ vertices are necessary for a minimal dominating set of D. If S_y does not contain x, then $S_y \cup \{y\}$ is a $\gamma(D)$ - set not containing x which is a contradiction. Hence the theorem.

Theorem 6
For any digraph D, $|V^0| \geq |V'|$.

Proof
For each $v \in V$, theorem 1.7establishes that for every γ - set S and $v \in S, pon\ [v, S]$ contains at least two vertices. These private neighbours of v are in $V - S$ and hence not in V'. Further from 1.8v is not adjacent to any vertex in V. Hence these private out nigh bourse are in V^0. Thus $|V^0| \geq 2|V'|$.

Theorem 7

For any digraph D, if $\gamma(D) \neq \gamma(D - v)$ for every $v \in V$, then $V' = V$.

Proof

Let $\gamma(D) \neq \gamma(D - v)$ for every $v \in V$. Then V' and V partition V. But if $v \in V$, then by theorem 1.6 V^0 is not empty, which is a contradiction. Hence $V' = V$.

Theorem 8

Let $D = (V, A)$ be a connected digraph. Then $\gamma(D - e) > \gamma(D)$ for every arc $e \in A(D)$ if and only if D is the oriented star K_{1v} oriented in such a way that all pendant vertices have outdegree 0.

Proof

If D is an oriented star K_{1v} oriented in such a way that all pendant vertices have outdegree 0, then $\gamma(D) = 1$ and $\gamma(D - e) = 2$ for every arc e in the digraph D.

Conversely suppose $\gamma(D - e) > \gamma(D)$ for every arc e in D. Let S be any γ - set of D. Clearly no arc can join two vertices of S or two vertices of $V - S$, since for any such arc e we have $\gamma(D - e) = \gamma(D)$. Also if $id(u) > 0)$ for any vertex u in S then for any arc of the form $e = (v, u)$, $\gamma(D - e) = \gamma(D)$ which is a contradiction. Hence $id\,(u) = 0$ for every $u \in S$. Suppose $|S| \geq$. Let $\{u_1, u_2\} \in S$. If there exists a vertex v in $V - S$ such that (u_1, v) and (u_2, v) are arcs in D then $\gamma(D - e) = \gamma(D)$ where $e = (u_1, v)$ which is a contradiction. Hence no two vertices in S have a common outneighbour. Since D is connected it follows that $|S| = 1$ and hence D is an oriented star K_{1n} oriented in such a way that all pendant vertices have out degree 0.

Theorem 9

If $D = (V, A)$ is any digraph with $\gamma(D - e) > \gamma(D)$ for every arc e in D, then every component of D is an oriented star oriented in such a way that all pendant vertices have outdegree 0.

We now consider a related problem. Bauer et al. [1] defined the bondage number of a graph to be the minimum number of edges whose removal increases the domination number. We consider the analogous concept for digraphs.

II. BONDAGE NUMBER

In a communications network, network consists of existing communication links between a fixed set of sites. The problem at hand is to select a smallest set of sites at which to place transmitters is joined by a direct communication link to one that does have a transmitter. This problem reduces to finding a minimum dominating set in the graph, corresponding to this network, that has a vertex corresponding to each site, and an edge between two vertices if and only if the corresponding sites have a direct communications link joining them.

Suppose that someone does not know which sites in the network act as transmitters, but does know that the set of such sites corresponds to a minimum dominating set in the related graph. What is the fewest number of communication links that he must sever so that at least one additional transmitter would be required in order that communication with all sites be possible? With this in mind, they introduce the bondage number of a graph

This concept was introduced by Fink et.al.[15] with the above application in mind. We now consider a related problem. Bauer et al. [4] defined the bondage number of a graph to be the minimum number of edges whose removal expands the domination number. We consider the analogous concept for digraphs.

Definition

The bondage number $b\,(D)$ of a digraph D is defined to be the minimum number of arcs whose removal increases the domination number of D.

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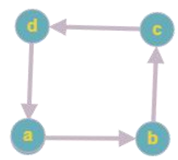
Since the domination number of every spanning subgraph of a digraph D is at least as great as $\gamma(D)$, the bondage number of a nonempty digraph is well defined.

Examples

(i) Consider the directed cycle $C_4 = (a,b,c,d,a)$ given in figure 7



C4

Fig 7

Clearly $\gamma(C_4) = 2$ and $\gamma(C_4 - e) = 2$ for every arc e of C_4. However $\gamma(C_4 - \{(a,b),(b,c)\}) = 3$. Hence $b(C_4) = 2$.

(ii) The bondage number of the directed tree given in figure 8 is 1.

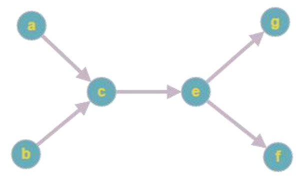


Fig 8

$\gamma(D) = 3$ and $\gamma(D - (e,f)) = 1$. Hence $b(D) = 1$

Theorem 1

Let D is a complete symmetric digraph on n vertices then $b(D) = n$.

Proof

Let D_1 be the subgraph obtained from D by removing n arcs of a hamiltoman cycle in D. Then the outdegree of every vertex is $n-2$ so that $\gamma(D_1) = 2 > \gamma(D)$. Hence $b(D) \le n$. Further if we remove any set of $n-1$ arcs from D then at most $n-1$ vertices of the resulting digraph will have outdegree less than $n-1$. Hence there exists at least one vertex in the resulting subdigraph D_2 having outdegree $n-1$. Hence $\gamma(D_2) = 1$. Thus $b(D) = n$.

III. CONLUSION

In this paper the effect of $\gamma(D)$ when D is modified by deletion of a vertex or deletion of an edge were investigated. It is very useful in the application of designing a graph model for fault tolerant computing system.

IV. ACKNOWLEDGEMENT

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REFERENCES

[1] C Berge, Theory of Graphs and its Applications, Methuen, London. (1962).

[2] C Berge, Graphs and Hypergraphs, North-Holland,Amsterdam. (1973).
[3] Changwoo Lee, On domination number of a digraph, Ph.D. Dissertation, Michigan State University (1994).
[4] D Bauer F.Harary, J. Nieminer and C.L. Suffel Domination alternation sets in graphs, Discrete Math., 47, (1983),153 – 161
[5] D. J.Kleitman and D. B. West, Spanning trees with many leaves. SIAM J. Discrete Math., 4, (1991), 99 - 106.
[6] E. A. Nordhausaand J. W.Gaddum, On complementary Graphs, Amer.Math.Monthly 63, (1956), 175 – 177
[7] E. J. Cockayne and S. T. Hedetniemi, Towards a theory of domination in grpahs, Networks, 7, (1977), 247 – 266.
[8] E.Sampathkumar and H. B. Walikar, The connected domination number of a graph, J. Math. Phys. Sci., 13, (1979), 607 -613.
[9] E.Samptahkumar and P. S. Neeralagi, Domination andneighbourhoodcritical, fixed, free and totally free points,Sankhya (Special Volume), 54, (1992), 403 - 405.
[10]F. Harary, Graph Theory, Addison - Wesley, Reading, MA,(1969)
[11]F. Jaeger ad C.Payan, Relations du type Nordhaus-Gaddumpour le nombre, d'absorption d'un graphe simple, C. R. Acad. Sci. Paris, 274, (1972), 728 – 730
[12]Haynes Teresa W. Stephen T. Hedetniemi, Peter J. Slater,Fundamentals of Domination in Graphs, Marcel Dekker, (1998).
[13]Haynes Teresa W., Stephen T. Hedetniemi, Peter J. Slater, Domination in Graphs Advanced Topics, Marcel Dekker, (1998)
[14]H.B.Walikar, B. D. Acharya and E. Sampathkumar, Recentdevelopments in the theory of domination in graphs, In MRILecture Notes in Math., Metha Research Inst., Allahabad,Volume I. (1979)
[15]J.F.Fink,M.J.Jacobson,L.F.Kinch,J.Roberts,Thebondagenumberofagraph,DiscreteMath.86(1990)47-57.
[16]J.PaulrajJosephandS.Arumugam, Domination in graphs,Internat. J. Management and Systems, 11, (1995), 177 - 185.
[17]J. R.Carrington, F.Harary and T. W. Haynes, Changing and unchanging the domination number of a graph, J. Combi.Math.CombinComput, 9, (1991), 57 – 63
[18]J. Von Neumann and O. Morgenstern, Theory of Games andEconomic behavior, Princeton Univ. Press, (1944).
[19]O. Ore. Theory of Graphs, Amer. Math. Soc. Colloq. Publ., 38(Amer.Math.Soc., Providence, RI), (1962).
[20]S. T. Hedetniemi, Hereditary properties of graphs, J. Combin Theory, 14, (1973), 16-27